STRESS STATE OF AN ECCENTRIC TUBE UNDER ELASTIC—PLASTIC STRAIN SUBJECTED TO PRESSURE

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UDC 539.374

The elastic—plastic state of an eccentric tube under the effect of internal pressure was examined in [1]. The solution was by the method of perturbations under the condition that the plastic zone completely encloses the internal contour. In this paper a modified perturbation method is proposed that will permit investigation of the stress state and the development of the plastic zone with only part of the contour enclosed. A problem on the elastic—plastic torsion of eccentric tubes [2] is solved by the method mentioned.

Let us consider the cross section of an eccentric tube (Fig. 1) from an ideal elastic—plastic material loaded by internal pressure. The equations of the inner and outer contours of the eccentric tube cross section have the form

$$\begin{gathered} L_1 : r = r_1, \\ L_2 : r = -\delta \cos\theta + \sqrt{1 - \delta^2 \sin^2\theta} = 1 - \delta\cos\theta + (\delta^2/2)\sin^2\theta + \ldots, \end{gathered} \tag{1}$$

where all the linear dimensions are referred to $r_2$.

In the absence of a plastic zone the internal pressure causes stress which is determined by the method of perturbations [3] with terms in $\delta$ not above $\delta^2$ taken into account:

$$\begin{gathered} \sigma_{r0} = a_0 r^{-2} + 2c_0 + \delta\left(2b_1 r - 2a_1' r^{-3}\right) \times \\ \times \cos\theta + \delta^2\left(a_0' r^{-2} + 2c_0' - \left(2a_2 + 6a_2' r^{-4} + 4b_2' r^{-2}\right)\cos 2\theta\right), \end{gathered} \tag{2}$$

$$\sigma_{\theta 0} = -a_0 r^{-2} + 2c_0 + \delta\left(6b_1 r + 2a_1' r^{-3}\right)\cos\theta + \delta^2\left(-a_0' r^{-2} + 2c_0' + \left(2a_2 + 12b_2 r^2 + 6a_2' r^{-4}\right)\cos 2\theta\right),$$

$$\tau_{r\theta 0} = \delta\left(2b_1 r - 2a_1' r^{-3}\right)\sin\theta + \delta^2\left(2a_2 - 6a_2' r^{-4} + 6b_2 r^2 - 2b_2' r^{-2}\right)\sin 2\theta,$$

where

$$a_0 = -\frac{pr_1^2}{1 - r_1^2}; \quad c_0 = \frac{pr_1^2}{2\left(1 - r_1^2\right)}; \quad a_1' = -\frac{a_0 r_1^4}{1 - r_1^4};$$

$$b_1 = -\frac{a_0}{1 - r_1^4}; \quad a_0' = \frac{Lr_1^2}{1 - r_1^2}; \quad c_0' = -\frac{L}{2\left(1 - r_1^2\right)};$$

$$a_2 = \frac{1}{2} D^{-1}\left(2G\left(r_1^{-2} - 1\right) - M\left(r_1^{-6} + r_1^{-2} - 2\right)\right);$$

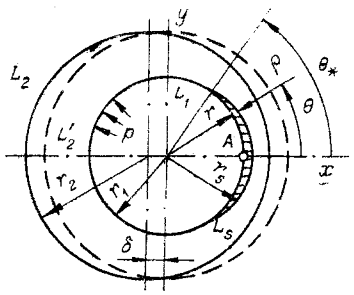


Fig. 1

Kuibyshev, Translated from Zhurnal Prikladnoi Mekhaniki i Tekhnicheskoi Fiziki, No. 3, pp. 152-159, May-June, 1983. Original article submitted May 7, 1982.

0021-8944/83/2403- 0429$07.50  

$$a_2' = \frac{1}{6} D^{-1} \left(2G\left(1 - r_1^2\right) - M\left(3r_1^{-2} - r_1^2 - 2\right)\right); \quad b_2 = \frac{1}{6} D^{-1} \left(G\left(r_1^{-6} - \right.\right.$$

$$\left.\left. - 4r_1^{-4} + 3r_1^{-2}\right) - M\left(-r_1^{-6} - 2r_1^{-4} + 3r_1^{-2}\right)\right); \quad b_2' = \frac{1}{2} D^{-1}\left(G\left(-r_1^{-2} + r_1^2\right) - \right.$$

$$\left. - M\left(-2r_1^{-4} + r_1^{-2} + r_1^2\right)\right); \quad L = \frac{4a_0}{1 - r_1^4}; \quad M = -\frac{a_0\left(3r_1^4 + 1\right)}{1 - r_1^4};$$

$$G = -\frac{a_0\left(5 + 3r_1^4\right)}{1 - r_1^4}; \quad D = r_1^{-6} - 4r_1^{-4} + 6r_1^{-2} + r_1^2 - 4.$$

The case is examined when the Poisson coefficient is $\mu = {}^1/_2$ and the Tresk and Mises plasticity conditions agree

$$(\sigma_\theta - \sigma_r)^2 + 4\tau_{r\theta}^2 = 4K^2 \tag{3}$$

(K is the shear yield point).

The case is investigated when the left side of (3) is maximal at the point A [4] and plastic flow starts from the point A when $p = p_0$. The value of $p_0$ is determined from (3) for $\theta = 0$ and $r = r_1$.

For $p > p_0$ the plastic zone will extend from the point A and will occupy a domain symmetric with respect to the x axis with an angle enclosing the inner contour $2\theta_*$. Henceforth, the angle enclosing the inner contour of the plastic zone is understood to be $\theta_*$.

The equation for the elastic–plastic boundary $L_s$ is written in the form

$$r_s = r_1 + \rho(p, \theta), \tag{4}$$

where $\rho(p, \theta)$ is the plastic zone thickness.

Let us introduce the smaller parameter

$$\varepsilon^2 = (p - p_0)/p_0. \tag{5}$$

The small parameter $\varepsilon$ from (5) is related to the angle enclosing the plastic zone $\theta_*$ of the contour $L_1$ by the relationship

$$\varepsilon \sim \theta_*, \quad \sin\theta_* \sim \varepsilon, \quad \cos\theta_* \sim 1. \tag{6}$$

We represent the stress in the elastic domain in the form

$$\sigma_r = \sum_{i=0}^{\infty} \sigma_{ri}\varepsilon^i, \; \sigma_\theta = \sum_{i=0}^{\infty} \sigma_{\theta i}\varepsilon^i, \; \tau_{r\theta} = \sum_{i=0}^{\infty} \tau_{r\theta i}\varepsilon^i. \tag{7}$$

According to [1] the stresses in the plastic domain have the form

$$\sigma_r^p = -p + 2K\ln\frac{r}{r_1}, \quad \sigma_\theta^p = -p + 2K\left(1 + \ln\frac{r}{r_1}\right). \tag{8}$$

We seek the function $\rho$ from (4) in the form of a series expansion in $\varepsilon$:

$$\rho = \sum_{k=1}^{\infty} \rho_k\varepsilon^k. \tag{9}$$

The stresses in the elastic domain are expressed in terms of the function $\varphi$ satisfying the biharmonic equation

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)\left(\frac{\partial^2\varphi}{\partial r^2} + \frac{1}{r}\frac{\partial\varphi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\varphi}{\partial\theta^2}\right) = 0, \tag{10}$$

where $\varphi$ is sought in the form $\varphi = \sum_{i=0}^{\infty} \varphi_i\varepsilon^i$.

It follows from (10)

$$\Delta^2\varphi_i = 0, \quad i = 0, 1, 2, \ldots \tag{11}$$

The stresses in each approximation have the form

$$\sigma_{ri} = \frac{1}{r}\frac{\partial\varphi_i}{\partial r} + \frac{1}{r^2}\frac{\partial^2\varphi_i}{\partial\theta^2}, \sigma_{\theta i} = \frac{\partial^2\varphi_i}{\partial r^2}, \quad \tau_{r\theta i} = \frac{1}{r^2}\frac{\partial\varphi_i}{\partial\theta} - \frac{1}{r}\frac{\partial^2\varphi_i}{\partial r\partial\theta}.$$

The boundary conditions on the contour $L_1$ are determined in the elastic domain from the condition of loading by internal pressure p with (5) and (7) taken into account

$$\sum_{i=0}^{\infty}\sigma_{ri}\varepsilon^i = -p_0(1+\varepsilon^2), \quad \sum_{i=0}^{\infty}\tau_{r\theta i}\varepsilon^i = 0. \tag{12}$$

Equating terms with identical powers of $\varepsilon$ in each of the equations in (12), we obtain boundary conditions for the different approximation on the contour $L_1$:

$$\sigma_{r0} = -p_0, \ \sigma_{r1} = 0, \ \sigma_{r2} = -p_0, \ \sigma_{ri} = 0, \ i = 3, 4, \ldots, \quad \tau_{r\theta i} = 0, \ i = 0, 1, 2, \ldots \tag{13}$$

The boundary conditions on the contour $L_2$ are:

$$\sum_{i=0}^{\infty}\sigma_{ri}\varepsilon^i\cos(n_2, r) + \sum_{i=0}^{\infty}\tau_{r\theta i}\varepsilon^i\cos(n_2, \theta) = 0, \quad \sum_{i=0}^{\infty}\sigma_{\theta i}\varepsilon^i\cos(n_2, \theta) + \sum_{i=0}^{\infty}\tau_{r\theta i}\varepsilon^i\cos(n_2, r) = 0, \tag{14}$$

where $(n_2, r)$, $(n_2, \theta)$ are angles between the normal to $L_2$ and the axes of a polar coordinate system. The expressions for $\cos(n_2, r)$ and $\cos(n_2, \theta)$ are:

$$\cos(n_2, r) = \frac{\Phi_{,r}}{\sqrt{\Phi_{,r}^2 + (r^{-1}\Phi_{,\theta})^2}}, \quad \cos(n_2, \theta) = \frac{r^{-1}\Phi_{,\theta}}{\sqrt{\Phi_{,r}^2 + (r^{-1}\Phi_{,\theta})^2}},$$

where $\Phi = r - 1 + \delta\cos\theta + (\delta^2/2)\sin^2\theta + \ldots$ is the equation of the outer contour of $L_2$. Substituting $\Phi_{,r}$, $\Phi_{,\theta}$ and manipulating, we obtain

$$\cos(n_2, r) = 1 - \delta^2\sin^2\theta/2 + \ldots, \cos(n_2, \theta) = -\delta\sin\theta + \ldots \tag{15}$$

Equating terms with identical powers of $\varepsilon$ in (14), we obtain boundary conditions on the contour $L_2$ for different approximations

$$\sigma_{ri}\cos(n_2, r) + \tau_{r\theta i}\cos(n_2, \theta) = 0, \quad \sigma_{\theta i}\cos(n_2, \theta) + \tau_{r\theta i}\cos(n_2, r) = 0, \ i = 0, 1, 2, \ldots \tag{16}$$

The boundary conditions for the elastic domain on the elastic–plastic boundary are determined from the condition of equal stresses:

$$\sigma_r = \sigma_r^p, \quad \sigma_\theta = \sigma_\theta^p, \quad \tau_{r\theta} = 0. \tag{17}$$

Using (4), (7), and (8), we expand (17) in a power series in $\rho$ with respect to the contour $L_1$ but limiting ourselves to terms with powers not above $\varepsilon^3$:

$$\sum_{i=0}^{\infty}\sum_{m=0}^{\infty}\frac{\varepsilon^i}{m!}\frac{\partial^m\sigma_{ri}}{\partial r^m}\left(\sum_{n=1}^{\infty}\rho_n\varepsilon^n\right)^m = \sum_{m=0}^{\infty}\frac{1}{m!}\frac{\partial^m\sigma_{r0}^p}{\partial r^m}\left(\sum_{n=1}^{\infty}\rho_n\varepsilon^n\right)^m, \tag{18}$$

and furthermore, expanding the sum

$$\sigma_{r0} + \frac{\partial \sigma_{r0}}{\partial r}(\varepsilon\rho_1 + \varepsilon^2\rho_2 + \varepsilon^3\rho_3) + \frac{\partial^2 \sigma_{r0}}{\partial r^2}\left(\frac{\varepsilon^2\rho_1^2}{2} + \varepsilon^3\rho_1\rho_2\right) +$$
$$+ \frac{\partial^3 \sigma_{r0}}{\partial r^3}\frac{\varepsilon^3\rho_1^3}{6} + \sigma_{r1}\varepsilon + \frac{\partial \sigma_{r1}}{\partial r}(\varepsilon^2\rho_1 + \varepsilon^3\rho_2) + \frac{\partial^2 \sigma_{r1}}{\partial r^2}\frac{\varepsilon^3\rho_1^2}{2} + \varepsilon^2\sigma_{r2} + \frac{\partial \sigma_{r2}}{\partial r}\varepsilon^3\rho_1 + \tag{19}$$
$$+ \varepsilon^3\sigma_{r3} = -p_0 - \varepsilon^2 p_0 + 2Kr_1^{-1}(\varepsilon\rho_1 + \varepsilon^2\rho_2 + \varepsilon^3\rho_3) - 2Kr_1^{-2}\left(\frac{\varepsilon^2\rho_1^2}{2} + \varepsilon^3\rho_1\rho_2\right) + \frac{2}{3}Kr_1^{-3}\varepsilon^3\rho_1^3;$$

$$\sum_{i=0}^{\infty}\sum_{m=0}^{\infty}\frac{\varepsilon^i}{m!}\frac{\partial^m \sigma_{\theta i}}{\partial r^m}\left(\sum_{n=1}^{\infty}\rho_n\varepsilon^n\right)^m = \sum_{m=0}^{\infty}\frac{1}{m!}\frac{\partial^m \sigma_{\theta 0}^p}{\partial r^m}\left(\sum_{n=1}^{\infty}\rho_n\varepsilon^n\right)^m; \tag{20}$$

$$\sum_{i=0}^{\infty}\sum_{m=0}^{\infty}\frac{\varepsilon^i}{m!}\frac{\partial^m \tau_{r\theta i}}{\partial r^m}\left(\sum_{n=1}^{\infty}\rho_n\varepsilon^n\right)^m = 0. \tag{21}$$

Upon expanding the sum in (20) we obtain an equality analogous to (19) with the difference that the right side equals zero. There follows from a comparison of the terms in (19)

$$2Kr_1^{-1} - \frac{\partial \sigma_{r0}}{\partial r} = 2\left(\delta B_{r1}\sin^2\frac{\theta}{2} + \delta^2 D_{r1}\sin^2\theta\right), \tag{22}$$

where $B_{r1} = 2b_1 + 6a_1' r_1^{-4}$ $D_{r1} = 24a_2' r_1^{-5} + 8b_2' r_1^{-3}$. After expanding $\sin^2(\theta/2)$ and $\sin^2\theta$ in series and retaining terms with $\theta$ not above $\theta^3$ we obtain

$$((1/2)\delta B_{r1} + 2\delta^2 D_{r1})\theta^2. \tag{23}$$

Taking account of (23) and (6) and equating terms for $\varepsilon$ in (19), we obtain the boundary conditions for $\sigma_{r1}$ on $L_1$

$$\sigma_{r1} = 0. \tag{24}$$

From (21) with (6) taken into account we obtain the boundary condition on $L_1$ for $\tau_{r\theta 1}$ after equating terms in $\varepsilon$:

$$\tau_{r\theta 1} = 0. \tag{25}$$

Taking account of (24) and (25), we have from the boundary conditions (13), (16), (17)

$$\varphi_1 = 0, \ \rho_1 = 0. \tag{26}$$

Taking account of (26), we obtain the problem to determine the second approximation stresses from (18) and (21):

$$\begin{gathered} \Delta^2\varphi_2;\ \sigma_{r2} = -p_0,\ \tau_{r\theta 2} = 0 \text{ on } L_1; \\ \sigma_{r2}\cos(n_2, r) + \tau_{r\theta 2}\cos(n_2, \theta) = 0, \\ \sigma_{\theta 2}\cos(n_2, \theta) + \tau_{r\theta 2}\cos(n_2, r) = 0 \text{ on } L_2. \end{gathered} \tag{27}$$

The boundary-value problem (27) agrees with the problem of elastic loading on an eccentric ring by internal pressure whose solution is (2). Keeping terms with $\theta$ not above $\theta^3$, it can be shown that

$$(-p_0 + 2K - \sigma_{\theta 0}) = L_1\theta^2, \tag{28}$$

where $L_1 = \frac{1}{2}\delta B_{\theta 0} + 2\delta^2 D_{\theta 0}$; $B_{\theta 0} = 6b_1 r_1 + 2a_1' r_1^{-3}$; $D_{\theta 0} = 2a_2 + 12b_2 r_1^2 + 6a_2' r_1^{-4}$.

From (20) we determine by taking (26), (27), (28) into account

$$\rho_2 = \frac{\sigma_{\theta 0}\varepsilon^2 - L_1\theta^2}{\left(-\frac{\partial \sigma_{\theta 0}}{\partial r} + 2Kr_1^{-1}\right)\varepsilon^2},$$

from which we obtain after substituting $\sigma_{\theta_0}$, $\partial\sigma_{\theta_0}/\partial r$ and taking (6) into account

$$\rho_2 = \frac{-L_1\theta^2 + M_1\varepsilon^2}{M_2\varepsilon^2}, \tag{29}$$

where

$$M_1 = A_{\theta 0} + \delta B_{\theta 0} + \delta^2 (C_{\theta 0} + D_{\theta 0}); \quad M_2 = -(A_{\theta 1} + \delta^2 (C_{\theta 1} + D_{\theta 1})) + 2Kr_1^{-1}; \quad A_{\theta 0} = -a_0 r_1^{-2} + 2c_0;$$
$$C_{\theta 0} = -a_0' r_1^{-2} + 2c_0'; \quad A_{\theta 1} = 2a_0 r_1^{-3}; \quad C_{\theta 1} = 2a_0' r_1^{-3}; \quad D_{\theta 1} = 24b_2 r_1 - 24a_2' r_1^{-5}.$$

The angle $\theta_*$ enclosing the contour $L_1$ of the plastic zone is determined from the condition $\rho_2 = 0$ for $\theta = \theta_*$:

$$\theta_* = \varepsilon \sqrt{M_1/L_1}. \tag{30}$$

$$\rho_2 = N\left(\theta_*^2 - \theta^2\right)/\varepsilon^2, \tag{31}$$

where $N = L_1/M_2$.

Let us determine the stress for the third approximation. We obtain the boundary conditions for the third approximation on $L_1$ by equating terms of identical powers with $\varepsilon^3$ from (18) and (21) with (31) taken into account

$$\sigma_{r3} = 0; \tag{32}$$

$$\tau_{r\theta 3} = -\frac{N\left(\theta_*^2 - \theta^2\right)}{\varepsilon^3}\frac{\partial \tau_{r\theta 0}}{\partial r}. \tag{33}$$

We substitute the value of $\partial\tau_{r\theta 0}/\partial r$ into (33), take account of (6) and obtain

$$\tau_{r\theta 3} = -T\left(\theta_*^2\theta - \theta^3\right)/\varepsilon^3, \tag{34}$$

where

$$T = (\delta E_1 + 2\delta^2 F_1) N, \quad E_1 = 2b_1 + 6a_1' r_1^{-4}, \quad F_1 = 24a_2' r_1^{-5} + 12b_2 r_1 + 4b_2' r_1^{-3}.$$

Let us determine the boundary conditions on $L_2$. The solution of the problem for the third approximation depends on the parameter $\delta$ which is the eccentricity of the tube. We seek the solution in the form of a power series in $\delta$:

$$\sigma_{r3} = \sum_{i=0}^{\infty} \sigma_{r3i}\delta^i, \quad \sigma_{\theta 3} = \sum_{i=0}^{\infty} \sigma_{\theta 3i}\delta^i, \quad \tau_{r\theta 3} = \sum_{i=0}^{\infty} \tau_{r\theta 3i}\delta^i. \tag{35}$$

It follows from (1) that the radii r of the contours $L_2$, $L_2'$ differ by $\Delta r = \delta\cos\theta + (\delta^2/2)\sin^2\theta + \ldots$ . After substituting (15) and (35) into (16) and expanding the stress in a power series in $\Delta r$ around the contour $L_2'$ disposed concentrically relative to $L_1$, we obtain

$$\sum_{m=0}^{\infty}\sum_{i=0}^{\infty}(-1)^m \frac{\partial^m \sigma_{r3i}}{\partial r^m}\frac{\left(\delta\cos\theta + \frac{\delta^2}{2}\sin^2\theta\right)^m}{m!}\delta^i\left(1 - \frac{\delta^2\sin^2\theta}{2} + \ldots\right) +$$
$$+ \sum_{m=0}^{\infty}\sum_{i=0}^{\infty}(-1)^m \frac{\partial^m \tau_{r\theta 3i}}{\partial r^m}\frac{\left(\delta\cos\theta + \frac{\delta^2}{2}\sin^2\theta\right)^m}{m!}\delta^i(-\delta\sin\theta + \ldots) = 0; \tag{36}$$

$$\sum_{m=0}^{\infty}\sum_{i=0}^{\infty}(-1)^m \frac{\partial^m \sigma_{\theta 3i}}{\partial r^m} \frac{\left(\delta\cos\theta + \frac{\delta^2}{2}\sin^2\theta\right)^m}{m!} \delta^i(-\delta\sin\theta + \ldots) +$$

$$+ \sum_{m=0}^{\infty}\sum_{i=0}^{\infty}(-1)^m \frac{\partial^m \tau_{r\theta 3i}}{\partial r^m} \frac{\left(\delta\cos\theta + \frac{\delta^2}{2}\sin^2\theta\right)^m}{m!} \delta^i\left(1 - \frac{\delta^2\sin^2\theta}{2} + \ldots\right) = 0. \tag{37}$$

We equate terms of identical powers in $\delta$ in (36) and (37) and limiting ourselves to terms with powers not higher than $\delta^2$ we obtain

$$\sigma_{r30} = 0, \quad \tau_{r\theta 30} = 0; \tag{38}$$

$$\sigma_{r31} = \frac{\partial\sigma_{r30}}{\partial r}\cos\theta, \quad \tau_{r\theta 31} = \sigma_{\theta 30}\sin\theta; \tag{39}$$

$$\sigma_{r32} = \frac{\sigma_{r30}\sin^2\theta}{2} + \frac{\partial\sigma_{r30}}{\partial r}\frac{\sin^2\theta}{2} - \frac{\partial^2\sigma_{r30}}{\partial r^2}\frac{\cos^2\theta}{2} + \frac{\partial\sigma_{r31}}{\partial r}\cos\theta + \tau_{r\theta 31}\sin\theta, \tag{40}$$

$$\tau_{r\theta 32} = -\frac{\partial\sigma_{\theta 30}}{\partial r}\sin\theta\cos\theta + \sigma_{\theta 31}\sin\theta + \frac{\partial\tau_{r\theta 31}}{\partial r}\cos\theta.$$

A solution of (11) under the boundary conditions (32) and (34) on $L_1$ and (36) and (37) on $L_2$ does not exist since the loads applied to the contour $L_1$ will not be equilibrated. Indeed, projections of the forces acting on the outer contour $L_2$ equal zero, while the forces acting on the inner contour $L_1$ equal

$$F_y = 0, \quad F_x = \int_{-\theta_*}^{\theta_*}(\sigma_{r3}\cos\theta - \tau_{r\theta 3}\sin\theta)\, r_1 d\theta. \tag{41}$$

From the integrand of (41) there follows that $\tau_{r\theta 3}\sin\theta$ is of the order of $\theta$. Hence, to obtain $F_x = 0$ such radial stresses are necessary as will, in their magnitude, be among approximations above the third. It therefore follows that radial stresses for the third approximation were excluded.

For a continuous assignment of $\tau_{r\theta 3}$ on the whole contour $L_1$ we expand the value of $\tau_{r\theta 3}$ (34) given in $-\theta_* \leqslant \theta \leqslant \theta_*$ in a Fourier series:

$$\tau_{r\theta 3} = \sum_{k=1}^{\infty} b_k \sin k\theta, \tag{42}$$

where

$$b_k = -\frac{2T}{\pi\varepsilon^3}\left(\left(-\frac{2\theta_*^2}{k^2} + \frac{6}{k^4}\right)\sin k\theta_* - \frac{6\theta_*}{k^3}\cos k\theta_*\right), \quad k = 1, 2, \ldots$$

It follows from (42) that only for k = 1 do the tangential stresses yield a force along the x axis. Hence, to solve the problem formulated, radial stresses should be introduced that equilibrate the tangential stresses for k = 1:

$$\sigma_{r3}^{(1)} = b_k\cos\theta, \quad k = 1, \tag{43}$$

which are of the order of $\theta$ or $\varepsilon$ higher than $\tau_{r\theta 3}$. Let us note that the tangential stresses will be self-equilibrated for $k \geqslant 2$.

The expressions (42) and (43) are the boundary conditions on $L_1$ for different k:

$$\sigma_{r3}^{(1)} = b_k\cos\theta, \quad \tau_{r\theta 3}^{(1)} = b_k\sin\theta, \quad k = 1; \tag{44}$$

$$\sigma_{r3}^{(k)} = 0, \quad \tau_{r\theta 3}^{(k)} = b_k\sin k\theta, \quad k \geqslant 2. \tag{45}$$

Let us limit ourselves to terms not higher than $\delta^2$. Equating terms of identical powers in $\delta$ into (44) and (45), we obtain boundary conditions on $L_1$ that correspond to (38)-(40) the boundary conditions on $L_2^!$:

$$\sigma_{r30}^{(k)} = 0, \quad \tau_{r\theta30}^{(k)} = 0, \qquad k = 0, 1, 2, \ldots; \tag{46}$$

$$\sigma_{r31}^{(k)} = 0, \quad \tau_{r\theta31}^{(k)} = 0, \qquad k = 0, 1, 2, \ldots; \tag{47}$$

$$\sigma_{r32}^{(1)} = e_k \cos\theta, \ \tau_{r\theta32}^{(1)} = e_k \sin\theta, \quad k = 1, \quad \sigma_{r32}^{(k)} = 0, \quad \tau_{r\theta32}^{(k)} = e_k \sin k\theta, \quad k = 2, 3, \ldots, \tag{48}$$

where

$$e_k = -\frac{8r_1^6}{\pi\varepsilon^3\left(1-r_1^4\right)^2\left(1-r_1^2\right)}\left(\left(-\frac{2\theta_*}{k^2}+\frac{6}{k^4}\right)\sin k\theta_* - \frac{6\theta_*}{k^3}\cos k\theta_*\right).$$

In conjunction with (46)-(48), expressions (38)-(40) are boundary conditions of the problem of tube loading by internal pressure proportional to sin kθ, cos kθ. Here (38) and (46) are the boundary conditions to determine the zeroth approximation stress in δ; (39) and (47) are to determine the first approximation stress in δ; and (40), (48) to determine the second approximation stress in δ.

Therefore, finding the third approximation stress reduces to solving the problem of loading a concentric tube by internal and external pressure proportional to sin kθ, cos kθ.

Using the solution in [5], we obtain

$$\sigma_{r3} = \delta^2\left(\sigma_{r32}^{(1)} + \sum_{k=2}^{\infty}\sigma_{r32}^{(k)}\right); \tag{49}$$

$$\sigma_{\theta3} = \delta^2\left(\sigma_{\theta32}^{(1)} + \sum_{k=2}^{\infty}\sigma_{\theta32}^{(k)}\right); \tag{50}$$

$$\tau_{r\theta3} = \delta^2\left(\tau_{r\theta32}^{(1)} + \sum_{k=2}^{\infty}\tau_{r\theta32}^{(k)}\right), \tag{51}$$

where

$$\sigma_{r32}^{(1)} = 2f\left(r - r^{-3}\right)\cos\theta; \quad \tau_{r\theta32}^{(1)} = 2f\left(r - r^{-3}\right)\sin\theta;$$

$$\sigma_{\theta32}^{(1)} = 2f\left(3r + r^{-3}\right)\cos\theta; \quad f = -\frac{e_k r_1^3}{2\left(1-r_1^4\right)}; \qquad k = 1;$$

$$\sigma_{r32}^{(k)} = \frac{1}{2R}\Big(\big(-(k-1)(k+2) + k^2\beta^2 + (k-2)\beta^{-2k}\big)q^{k-2} + \big((k-2)(k+1) + (k+2)\beta^{2k} - k^2\beta^2\big)q^{-(k+2)} + \big(-(k-2)(k+1) + (k^2-4)\beta^{-2} - (k-2)\beta^{-2k}\big)q^k + \big((k-1)(k+2) - (k^2-4)\beta^{-2} - (k+2)\beta^{2k}\big)q^{-k}\Big)e_k\cos k\theta;$$

$$\sigma_{\theta31}^{(k)} = \frac{1}{2R}\Big(\big((k-1)(k+2) - k^2\beta^2 - (k-2)\beta^{-2k}\big)q^{k-2}\big) + \big(-(k-2)(k+1) + k^2\beta^2 - (k+2)\beta^{2k}\big)q^{-(k+2)} + \big((k+1)(k+2) - (k+2)^2\beta^{-2} + (k+2)\beta^{-2k}\big)q^k + \big(-(k-2)(k-1) + (k-2)^2\beta^{-2} + (k-2)\beta^{2k}\big)q^{-k}\Big)e_k\cos k\theta;$$

$$\tau_{r\theta31}^{(k)} = \frac{1}{2R}\Big(\big((k-1)(k+2) - k^2\beta^2 - (k-2)\beta^{-2k}\big)q^{k-2} + \big((k-2)(k+1) - k^2\beta^2 + (k+2)\beta^{2k}\big)q^{-(k+2)} + \big(k(k+1) - k(k+2)\beta^{-2} + k\beta^{-2k}\big)q^k + \big(k(k-1) - k(k-2)\beta^{-2} - k\beta^{2k}\big)q^{-k}\Big)e_k\sin k\theta;$$

$$R = 2(k^2-1) - k^2\left(\beta^{-2} + \beta^2\right) + \left(\beta^{-2k} + \beta^{2k}\right); \quad \beta = \frac{1}{r_1}, \quad q = \frac{r}{r_1}.$$

Having obtained the value $\sigma_{\theta3}$ (50), we determine $\rho_3$ from (20) by equating terms with $\varepsilon^3$:

$$\rho_3 = \frac{\sigma_{\theta3}}{2Kr_1^{-1} - \frac{\partial\sigma_{\theta0}}{\partial r}}. \tag{52}$$

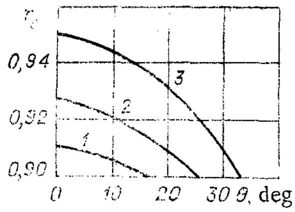


Fig. 2

According to (9), by using (29) and (52) we obtain the plastic zone thickness

$$\rho = \varepsilon^2\rho_2 + \varepsilon^3\rho_3$$

and from (4) we obtain the position of the elastic—plastic boundary (see Fig. 1).

We obtain the stresses in an elastic domain from (7), (26), (27), (49)-(51) by limiting ourselves to the third approximation

$$\sigma_r = (1 + \varepsilon^2)\sigma_{r0} + \varepsilon^3\sigma_{r3},\ \sigma_\theta = (1 + \varepsilon^2)\sigma_{\theta 0} + \varepsilon^3\sigma_{\theta 3},\ \tau_{r\theta} = (1 + \varepsilon^2)\tau_{r\theta 0} + \varepsilon^3\tau_{r\theta 3}.$$

The solution obtained satisfies the exact equations of the theory of ideal plasticity in the plastic domain and the theory of elasticity in the elastic domain. Here because of the limited number of approximations, the boundary conditions on the outer contour and the conjugate conditions on the elastic—plastic boundary are satisfied approximately.

Therefore the accuracy of the solution obtained can be determined with resepct to the residual on the elastic—plastic boundary $1 - \sqrt{(\sigma_\theta - \sigma_r)^2 + 4\tau_{r\theta}^2}(2K)^{-1}$ and with respect to the relative boundary conditions on the outer contour $p_\nu(2K)^{-1}$, where

$$p_\nu = \sqrt{\sigma^2 + \tau^2};\ \sigma = \sigma_r \cos^2(n_2, r) + \sigma_\theta \cos^2(n_2, \theta) + 2\tau_{r\theta} \cos(n_2, r) \cos(n_2, \theta);$$

$$\tau = (\sigma_\theta - \sigma_r) \cos(n_2, r) \cos(n_2, \theta) + \tau_{r\theta}(\cos^2(n_2, r) - \cos^2(n_2, \theta)).$$

The location of the boundaries $L_s$ is represented in Fig. 2 for a tube with the parameters $r_1/r_2 = 0.9$; $\delta/r_2 = 0.05$ for $\varepsilon = 0.13$; 0.22; 0.29 (curves 1-3), respectively.

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